Exam Analysis on Manifolds

Wednesday, April 10, 2013, 14:00–17:00

This exam consists of three assignments. You get 10 points for free. Using notes or books is not allowed.

Problem 1. If $x = (x^1, x^2)$ is a nonzero point in $\mathbb{R}^2$, then the straight line passing through that point and the origin is the set $\{\lambda(x^1, x^2) \mid \lambda \in \mathbb{R}\}$. Denote this line by $[x^1, x^2]$. The set of *all* straight lines in $\mathbb{R}^2$ through the origin is denoted by $\mathbb{P}^1 = \{[x^1, x^2] \mid (x^1, x^2) \in \mathbb{R}^2 \setminus \{(0,0)\}\}$, and is called the *real projective line*. If $[x^1, x^2] \in \mathbb{P}^1$ then x^1, x^2 are called *homogeneous coordinates*.

Note that if $(x^1, x^2) = \lambda(y^1, y^2)$ then $[x^1, x^2] = [y^1, y^2]$. However, if $[x^1, x^2]$ is such that $x^1 \neq 0$, and $[y^1, y^2] = [x^1, x^2]$, then y^1 is also nonzero. Thus, denote with $U_1 \subset \mathbb{P}^1$ the subset of elements whose first homogeneous coordinate is nonzero. Similarly, $U_2 \subset \mathbb{P}^1$ is the subset whose second homogeneous coordinate is nonzero.

We define two continuous maps $\varphi_1 \colon U_1 \to \mathbb{R}$ and $\varphi_2 \colon U_2 \to \mathbb{R}$ by

$$\varphi_1\big([x^1, x^2]\big) = \frac{x^2}{x^1}, \qquad \varphi_2\big([x^1, x^2]\big) = \frac{x^1}{x^2}.$$

(a) (5 points). Show that these maps do not depend on the homogeneous coordinates x^1 and x^2 of $[x^1, x^2]$; i.e., if $[x^1, x^2] = [y^1, y^2]$ then $\varphi_i([x^1, x^2]) = \varphi_i([y^1, y^2])$ for $i = 1, 2$.

(b) (10 points). Show that the continuous maps $\psi_i \colon \mathbb{R} \to \mathbb{P}^1$ defined by

$$\psi_1(u) = [1, u], \qquad \psi_2(u) = [u, 1]$$

are left- and right-inverses for φ_1 and φ_2, respectively.

(c) (10 points.) Show that $\mathscr{A} = \{(U_1, \varphi_1), (U_2, \varphi_2)\}$ is a smooth atlas for $\mathbb{P}^1$.

Solution.

(a) If $[x^1, x^2] = [y^1, y^2]$, then any element from the one is an element of the other. In particular, $(y^1, y^2) \in [x^1, x^2]$ which means $(y^1, y^2) = \lambda(x^1, x^2)$ for some λ, and then

$$\varphi_2([y^1, y^2]) = \frac{y^1}{y^2} = \frac{\lambda x^1}{\lambda x^2} = \frac{x^1}{x^2} = \varphi_1([x^1, x^2]).$$

Similarly for φ_2. (Note that since y^1, y^2 are homogeneous coordinates for $[y^1, y^2]$, they are not simultaneously zero, so that λ also must be nonzero.)

(b) We calculate

$$\varphi_1(\psi_1(u)) = \varphi_1([1, u]) = \frac{u}{1} = u,$$
$$\psi_1(\varphi_1([x^1, x^2])) = \psi_1\left(\frac{x^2}{x^1}\right) = \left[1, \frac{x^2}{x^1}\right] = [x^1, x^2].$$

Similarly for ψ_2.

(c) U_1 and U_2 cover $\mathbb{P}^1$, because $[0,0] \notin \mathbb{P}^1$, and since $\varphi_i^{-1} = \psi_i$ and both are continuous, they are in fact homeomorphisms and therefore charts. Lastly, if $[x^1, x^2] \in U_1 \cap U_2$ then the transition map $\kappa\colon \varphi_1(U_1 \cap U_2) = \mathbb{R} \setminus 0 \to \varphi_2(U_1 \cap U_2)$ is

$$\kappa(u) = \varphi_2 \circ \varphi_1^{-1}(u) = \varphi_2([1,u]) = \frac{1}{u},$$

which is clearly smooth (note that u cannot be zero). The same obviously holds for the other transition map. Therefore this atlas is a smooth atlas.

Problem 2. (25 points.) Consider $\mathbb{R}^3$ with coordinates (x,y,z) and $\mathbb{R}^2$ with coordinates (θ,φ). Let $g\colon \mathbb{R}^2 \to \mathbb{R}^3$ be given by $g(\theta,\varphi) = (\cos\varphi\cos\theta, \cos\varphi\sin\theta, \sin\varphi)$, and let $\omega = xy\,\mathrm{d}z \in \Omega^1(\mathbb{R}^3)$. Calculate $\mathrm{d}(g^*\omega)$ and $g^*(\mathrm{d}\omega)$ separately, and verify that they are equal.

Solution. We calculate

$$\begin{aligned}
g^*(\omega) &= g^*(xy\,\mathrm{d}z) = \big((xy)\ \circ g\big)\,g^*(\mathrm{d}z) = (x\circ g)(y\circ g)\,\mathrm{d}(z\circ g)\\
&= \cos\varphi\cos\theta\cos\varphi\sin\theta\,\mathrm{d}(\sin\varphi) = \cos^3\varphi\cos\theta\sin\theta\,\mathrm{d}\varphi,\\
\mathrm{d}(g^*\omega) &= \mathrm{d}(\cos^3\varphi\cos\theta\sin\theta\,\mathrm{d}\varphi) = \frac{\partial}{\partial\theta}(\cos^3\varphi\cos\theta\sin\theta)\,\mathrm{d}\theta\wedge\mathrm{d}\varphi\\
&= \cos^3\varphi(\cos^2\theta - \sin^2\theta)\,\mathrm{d}\theta\wedge\mathrm{d}\varphi.
\end{aligned}$$

On the other hand

$$\begin{aligned}
\mathrm{d}\omega &= x\,\mathrm{d}y\wedge\mathrm{d}z + y\mathrm{d}x\wedge\mathrm{d}z,\\
g^*(\mathrm{d}\omega) &= (x\circ g)\,\mathrm{d}(y\circ g)\wedge(z\circ g) + (y\circ g)\,\mathrm{d}(x\circ g)\wedge(z\circ g)\\
&= \cos\varphi\cos\theta\frac{\partial}{\partial\theta}(\cos\varphi\sin\theta)\cos\varphi\,\mathrm{d}\theta\wedge\mathrm{d}\varphi + \cos\varphi\sin\theta\frac{\partial}{\partial\theta}(\cos\varphi\cos\theta)\cos\varphi\,\mathrm{d}\theta\wedge\mathrm{d}\varphi\\
&= \cos^3\varphi\cos^2\theta\,\mathrm{d}\theta\wedge\mathrm{d}\varphi - \cos^3\varphi\sin^2\theta\,\mathrm{d}\theta\wedge\mathrm{d}\varphi = g^*(\omega).
\end{aligned}$$

10 points for each side.

Problem 3. Let $\mathbb{B}^n = \{x \in \mathbb{R}^n \mid |x|^2 = (x^1)^2 + \cdots + (x^n)^2 \le 1\}$ be the filled sphere, and let

$$\omega = \sum_{i=1}^{n}(-1)^{i-1}x^i\,\mathrm{d}x^1\wedge\cdots\wedge\widehat{\mathrm{d}x^i}\wedge\cdots\wedge\mathrm{d}x^n \in \Omega^{n-1}(\mathbb{B}^n)$$

(the hat over $\mathrm{d}x^i$ in the i-th term means that it is *not* present in that term).

(a) (10 points.) Calculate $\mathrm{d}\omega$.

(b) (10 points.) Show that

$$\int_{\partial\mathbb{B}^n}\omega > 0.$$

(c) (10 points.) Restricting ω to $\partial\mathbb{B}^n$ yields an $(n-1)$-form $\omega|_{\partial\mathbb{B}^n} \in \Omega^{n-1}(\partial\mathbb{B}^n)$. Suppose $g\colon \mathbb{B}^n \to \partial\mathbb{B}^n$ is a smooth map from $\mathbb{B}^n$ to its boundary. Show that

$$\mathrm{d}\big(g^*(\omega|_{\partial\mathbb{B}^n})\big) = 0.$$

(d) (10 points). Restricting g to the boundary $\partial\mathbb{B}^n$ of $\mathbb{B}^n$, we obtain a map $g|_{\partial\mathbb{B}^n} : \partial\mathbb{B}^n \to \partial\mathbb{B}^n$. Show that $g|_{\partial\mathbb{B}^n}$ is not the identity.

Solution.

(a) $\mathrm{d}\omega = n\,\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$.

(b) $\int_{\partial\mathbb{B}^n} \omega = \int_{\mathbb{B}^n} \mathrm{d}\omega = n \int_{\mathbb{B}^n} \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n = n\,\mathrm{Vol}(\mathbb{B}^n) > 0.$

(c) We know $\mathrm{d}(g^*\omega|_{\partial\mathbb{B}^n}) = g^*(\mathrm{d}\omega|_{\partial\mathbb{B}^n})$, but then $\mathrm{d}\omega|_{\partial\mathbb{B}^n}$ is is an n-form on an $(n-1)$-dimensional manifold. Therefore it is 0.

(d) Suppose that this restriction is the identity, then, using (b),

$$0 < \int_{\partial\mathbb{B}^n} \omega = \int_{\partial\mathbb{B}^n} \mathrm{id}^*\,\omega = \int_{\partial\mathbb{B}^n} g^*\omega = \int_{\mathbb{B}^n} \mathrm{d}(g^*\omega) = 0,$$

a contradiction.